Role of fluctuations in the response of coupled bistable units to weak time-periodic driving forces

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We analyze the stochastic response of a finite set of globally coupled noisy bistable units driven by rather weak time-periodic forces. We focus on the stochastic resonance and phase frequency synchronization of the collective variable, defined as the arithmetic mean of the variable characterizing each element of the array. For single-unit systems, stochastic resonance can be understood with the powerful tools of linear response theory. Proper noise-induced phase frequency synchronization for a single-unit system in this linear response regime does not exist. For coupled arrays, our numerical simulations indicate an enhancement of the stochastic resonance effects leading to gains larger than unity as well as genuine phase frequency synchronization. The nonmonotonicity of the response with the strength of the coupling strength is investigated. Comparison with simplifying schemes proposed in the literature to describe the random response of the collective variable is carried out.

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I. INTRODUCTION

The response of noisy nonlinear systems driven by weak external driving terms is frequently analyzed with the powerful tools of linear response theory (LRT) [1,2]. The amplification of the amplitude of the average response with the noise strength as well as the nonmonotonic behaviors of the amplitude and of the output signal-to-noise ratio are manifestations of stochastic resonance (SR) [3]. These behaviors have been indeed rationalized using the ideas of LRT [4]. Within the linear regime described by LRT, the response of the nonlinear system is necessarily very noisy. The output noise level is much higher and the decay time of the fluctuations much longer than those present in a linear system subject to the same noise and driving terms. Consequently, the signal-to-noise ratio of the nonlinear system in that regime is very small, even at its peak value, and the gain does not exceed unity [5]. If one is interested in obtaining large average amplitudes and large signal-to-noise ratio, one must find a way to control the output fluctuations and this is only possible in a nonlinear regime.

A possible way to reduce the output fluctuation levels in noise-induced SR is to use external driving terms with amplitudes so large that, even though they are still subthreshold, they alter significantly the potential relief of the dynamics and render invalid the LRT assumptions [6]. Another possibility is to concentrate on the global response of a set of nonlinear oscillators. In [7–10] the enhancement of SR effects in arrays of linearly or nonlinearly globally coupled arrays of bistable systems were studied. Array-enhanced stochastic resonance effects for an oscillator coupled to an array of locally coupled identical oscillators have also been reported in [11]. More recently [12,13], we have analyzed the *collective* response of a finite set of globally coupled bistable systems. We demonstrated that SR is much enhanced with respect to SR in single bistable units. Indeed, gains larger than unity were observed for subthreshold sinusoidal driving forces. Those findings indicated that the arrays were indeed operating in nonlinear regimes.

Another aspect of the stochastic response of the system refers to its noise-induced synchronization with the driving term. By a suitable definition of a phase associated to the random output process, one can define an output average phase frequency and a phase dispersion. The matching of the phase frequency with the driving frequency for a range of noise values is what is termed noise-induced frequency synchronization. It has been analyzed with a variety of analytical and numerical procedures [14–17], even in circumstances where quantum tunneling effects are relevant [18]. In [19] noise-induced phase synchronization of the collective variable for wide ranges of noise values was also observed when the array was driven by subthreshold inputs.

The results reported in the above-mentioned work were obtained with driving terms that, even though they were unable to produce the cited effects in the absence of noise, they were large enough to invalidate LRT when applied to a single bistable system. The question we address in the present work is whether the enhancement of SR and synchronization effects in the collective response of finite arrays of bistable units still persist when the driving external force is rather weak. By weak, we will mean here that (i) the SR effects induced by the external driving in a single bistable unit can be satisfactorily described, even at a quantitative level, by the LRT approximation; and (ii) noise-induced phase frequency synchronization in a single bistable unit does not properly exist.

In the next section, we introduce the model system and define the relevant quantities that characterize the phenomenon of SR and phase synchronization. In Sec. III we show the results obtained by numerically solving the dynamical stochastic equations. Comparison with the predictions of recently formulated approximate descriptions of the collective dynamics [20,21] is carried out in Sec. IV. The last section concludes with some remarks.

II. MODEL AND DEFINITIONS

We consider a set of N identical subsystems, each of them characterized by a variable $x_i(t)$ $(i=1,\dots,N)$ satisfying a sto-

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chastic evolution equation (in dimensionless form) of the type [12,20,22]

$$\dot{x}_i = x_i - x_i^3 + \frac{\theta}{N}\sum_{j=1}^{N}(x_j - x_i) + \sqrt{2D}\xi_i(t) + F(t). \tag{1}$$

The external driving force is periodic in t, $F(t)=F(t+T)$. The term $\xi_i(t)$ represents a white noise with zero average and $\langle \xi_i(t)\xi_j(s)\rangle = \delta_{ij}\delta(t-s)$. We define a collective variable $S(t)$,

$$S(t) = \frac{1}{N}\sum_j x_j(t), \tag{2}$$

and concentrate on its long-time response when the system size N is kept finite and the amplitude of the driving term is weak in the sense indicated in Sec. I. We define the one-time correlation function

$$L(\tau) = \frac{1}{T}\int_0^T dt \langle S(t)S(t+\tau)\rangle_*. \tag{3}$$

The notation $\langle \cdots \rangle$ indicates an average over the noise realizations and the subscript asterisk indicates the long-time limit of the noise average, i.e., its value after waiting for t large enough that transients have died out. As shown in our previous work [12], we have that

$$L(\tau) = L_{\text{coh}}(\tau) + L_{\text{incoh}}(\tau), \tag{4}$$

where the coherent part $L_{\text{coh}}(\tau)$ is periodic in τ with the period of the driving force, while the incoherent part $L_{\text{incoh}}(\tau)$ arising from the fluctuations of the output $S(t)$ around its average value, decays to zero as τ increases. The output signal-to-noise ratio (SNR) R_{out} is

$$R_{\text{out}} = \lim_{\epsilon\to 0^+} \frac{\int_{\Omega-\epsilon}^{\Omega+\epsilon} d\omega\, \tilde{L}(\omega)}{\tilde{L}_{\text{incoh}}(\Omega)} = \frac{\tilde{L}_{\text{coh}}(\Omega)}{\tilde{L}_{\text{incoh}}(\Omega)}, \tag{5}$$

where Ω is the fundamental frequency of the driving force $F(t)$, $\tilde{L}_{\text{coh}}(\Omega)$ is the corresponding Fourier coefficient in the Fourier series expansion of $L_{\text{coh}}(\tau)$, and $\tilde{L}_{\text{incoh}}(\Omega)$ is the Fourier transform at frequency Ω of $L_{\text{incoh}}(\tau)$.

For a set of N coupled linear oscillators driven by an external driving force $F(t)$ and subject to the noise terms $\xi_i(t)$ as in Eq. (1), the SNR of the corresponding collective process, $R_{\text{out}}^{(L)}$, coincides with that of the random process formed by the arithmetic mean of the individual noise terms $\xi_i(t)$ plus the deterministic driving force $F(t)$, namely, $F(t)+\xi(t)$ with $\xi(t)=N^{-1}\Sigma_{i=1}^{N}\xi_i(t)$. The process $\xi(t)$ is a Gaussian white noise of effective strength D/N. In this work, we have considered an external periodic rectangular driving,

$$F(t) = (-1)^{n(t)}A, \tag{6}$$

where $n(t)=\lfloor 2t/T \rfloor$; $\lfloor z \rfloor$ is the floor function of z, i.e., the greatest integer less than or equal to z. In other words, $F(t)=A$ $[F(t)=-A]$ if $t\in[nT/2,(n+1)T/2]$ with n even (odd). Then it is easy to prove that

$$R_{\text{out}}^{(L)} = \frac{4A^2N}{\pi D}. \tag{7}$$

Thus, for our nonlinear case, it seems convenient to analyze the SR gain G defined as [12]

$$G = \frac{R_{\text{out}}}{R_{\text{out}}^{(L)}}, \tag{8}$$

which compares the SNR of a nonlinear system with that of a linear system subject to the same stochastic and deterministic forces.

In the case of noninteracting units ($\theta=0$), the SNR of the collective output is N times larger than that of a isolated unit driven by the same force. Nonetheless, as discussed in [12], the gain associated with the collective output is just the same as the one of a single, isolated, unit. Thus, for the weak forces that we are considering here, we expect that the collective gain will not exceed unity. As seen below, our numerical results will confirm that expectation.

Another aspect of the response is the noise induced phase frequency synchronization. We note that for low noise strengths and driving forces with sufficiently large periods, a random trajectory of $S(t)$ contains essentially small fluctuations around two values (attractors) and random, sporadic transitions between them. For each realization of the noise term, we then introduce a random phase process $\phi(t)$ associated with the stochastic variable $S(t)$ as follows. We refer to a "jump" of $S(t)$ along a trajectory, when a very large fluctuation takes the $S(t)$ trajectory from a value near an attractor to a value in the neighborhood of the other attractor. We count $N^{(\alpha)}(t)$, the number of jumps in the α trajectory of the process $S(t)$ within the interval $(0,t]$. A trajectory of the phase process is then constructed as

$$\phi^{(\alpha)}(t) = \pi N^{(\alpha)}(t), \tag{9}$$

so that $\phi(t)$ increases by 2π after every two consecutive jumps.

The first two moments of the phase process are estimated as

$$\langle \phi(t)\rangle = \frac{1}{\mathcal{M}}\sum_{\alpha=1}^{\mathcal{M}} \phi^{(\alpha)}(t), \tag{10}$$

$$v(t) = \langle [\phi(t)]^2\rangle - \langle \phi(t)\rangle^2 = \frac{1}{\mathcal{M}}\sum_{\alpha=1}^{\mathcal{M}}[\phi^{(\alpha)}(t)]^2 - \frac{1}{\mathcal{M}^2}\left(\sum_{\alpha=1}^{\mathcal{M}}\phi^{(\alpha)}(t)\right)^2 \tag{11}$$

where $\mathcal{M}$ is the number of generated random trajectories.

The instantaneous phase frequency is easily determined from the time derivative of $\langle \phi(t)\rangle$. After a sufficiently large number of periods of the driving force, n, the system forgets its initial preparation, but the instantaneous phase frequency is still a function of time. Then, we define a cycle average phase frequency $\bar{\Omega}_{\text{ph}}$ by averaging the instantaneous phase frequency over a period of the external driving [18,23]

$$\bar{\Omega}_{\rm ph} = \frac{1}{T}\int_{nT}^{(n+1)T} dt \frac{d\langle \phi(t)\rangle}{dt} = \frac{\langle \phi[(n+1)T]\rangle - \langle \phi(nT)\rangle}{T}. \tag{12}$$

Similarly, the cycle average phase diffusion coefficient is evaluated from the instantaneous slope of the variance $v(t)$ as [18,23]

$$\bar{D}_{\rm ph} = \frac{1}{T}\int_{nT}^{(n+1)T} dt \frac{d\langle v(t)\rangle}{dt} = \frac{v[(n+1)T] - v(nT)}{T}. \tag{13}$$

In previous works, approximate analytical expressions for these two quantities have been derived for the $N=1$ problem in the classical [23–26] and quantum cases [18]. Those expressions can not be applied to the collective variable of an N-particle problem,

III. NUMERICAL RESULTS

In general, nonlinearities preclude exact analytical solutions of Eqs. (1). We will use numerical simulations to obtain useful information about the stochastic process $S(t)$. In the asymptotic limit $N\to\infty$, Desai and Zwanzig [22] showed that the statistical properties of the model could be analyzed in terms of a nonlinear Fokker-Planck equation which allows the coexistence of several stable probability distributions for some ranges of noise strengths and coupling constants. In the same asymptotic limit, we analyzed a few years ago the stochastic resonant behavior of the first moment $\langle S(t)\rangle_*$ when the system is driven by a time-dependent sinusoidal force, using a combination of analytical and numerical procedures [9]. In particular, for very weak input amplitudes, a linear response theory analysis showed that a huge amplification in the amplitude of the average output $\langle S(t)\rangle_*$ with respect to that of the driving force could be achieved.

In this work we concentrate on situations where (i) the number of subunits is finite; (ii) the amplitude of the driving force is rather weak. As detailed in [6] the numerical algorithm used to integrate the Langevin equations follows one of the schemes put forward by Greenside and Helfand [27].

Let us first consider the case of independent subunits, $\theta=0$. In Fig. 1 we display the behavior with respect to the noise strength D of the signal-to-noise ratio and the gain of the output variable $S(t)$ for an array of $N=10$ noninteracting bistable units, driven by a weak amplitude rectangular force ($A=0.1$) with fundamental frequency $\Omega=0.01$. For comparison purposes, in Fig. 2 we present the results for the signal-to-noise ratio and the gain of the response of a single bistable units operating under the action of the same driving term as in the previous figure Clearly, SR is manifested in the nonmonotonic behavior of the signal-to-noise ratio with respect to the noise strength in both figures. But, as expected, the R values for the $N=1$ case are very small, $1/10$ times the R values for the $N=10$ case. On the other hand, the gain in both cases has the same values, not exceeding unity as required by the linear response theory. The expected amplification of the R values are in agreement with the predictions of the central limit theorem which indicates a reduction by an $1/N$ factor

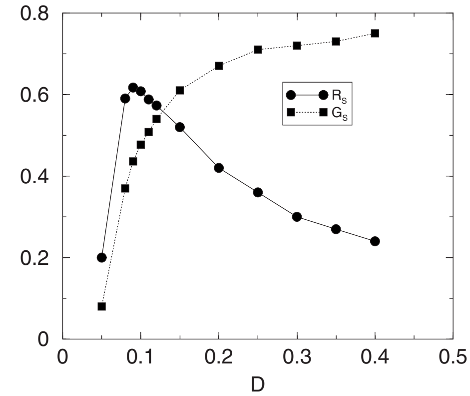

FIG. 1. Signal-to-noise ratio R_S (circles) and gain G_S (squares) of the collective variable for an array of $N=10$ noncoupled, ($\theta=0$), bistable units driven by rectangular inputs with $\Omega=0.01$ and $A=0.1$.

of the output fluctuations of the array with respect to those of a system with a single unit.

The interactions between the units bring up changes in the collective response as depicted in Fig. 3. Here we have coupled the $N=10$ bistable elements with a weak coupling strength $\theta=0.2$. The existence of coupling increases substantially the nonmonotonic behavior of R vs D of the collective output relative to the uncoupled units case in Fig. 1. Also, the peak value is reached at higher values of D. And, more importantly, the gain is clearly above unity for a wide range of noise values. This fact indicates that the array is operating in a nonlinear regime even though the driving amplitude A is rather small.

The dependence on the interaction strength θ of R and G is explicitly demonstrated in the next figure, Fig. 4. Here, we depict the peak values of the signal-to-noise ratio $R_{\max}$ and the gain $G_{\max}$ on the coupling strength θ for an array of $N=10$ elements driven by a rectangular input with parameters $\Omega=0.01$ and $A=0.1$. The nonmonotonic behavior with θ is clear. As the coupling strength increases from zero the SR effects become more pronounced until they reach a maximum at around $\theta=0.5$. Increasing further the coupling strength leads to a decrease in the peak values.

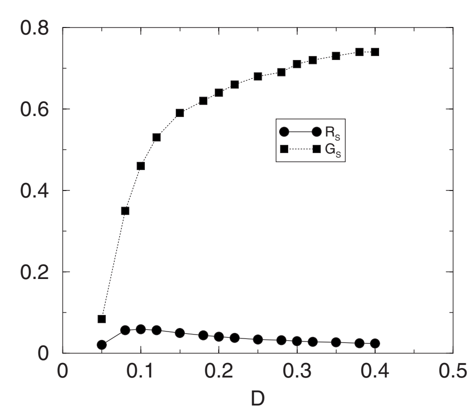

FIG. 2. Signal-to-noise ratio R_S (circles) and gain G_S (squares) for a single bistable unit driven by rectangular inputs with $\Omega=0.01$ and $A=0.1$.

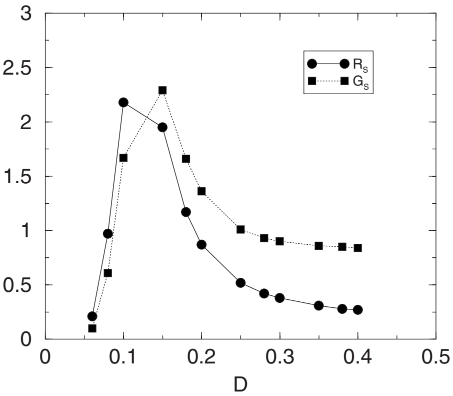

FIG. 3. Signal-to-noise ratio R_S (circles) and gain G_S (squares) of the collective variable for an array of $N=10$ bistable units driven by rectangular inputs with $\Omega=0.01$ and $A=0.1$ and coupling strength $\theta=0.2$.

To understand the nonmonotonic behavior of $R_{\max}$ with θ depicted in the figure above, it is relevant to study the dependence on θ of the two components of the correlation function: its coherent and incoherent parts. In the lower panel in Fig. 5, we depict the maximum value of the coherent part of the correlation function (C_{coh}) on the coupling strength θ for those noise values at which the amplification is maximal. In the upper panel, the dependence of the initial value of the corresponding incoherent part $C_{\text{incoh}}(0)$ with θ is shown. One observes that both quantities show nonmonotonic behaviors with the coupling strength. Even though the peak in the lower panel and the minimum in the higher panel are obtained for $\theta=0.2$, the largest signal-to-noise ratio is reached at the slightly larger coupling strength $\theta=0.5$. This can be understood by noting that, as depicted in Fig. 6, the decay of the incoherent part of the correlation function at $\theta=0.5$ is faster than at the other coupling values. Indeed, the key behavior is that of the incoherent part of the correlation function. Comparing $C_{\text{incoh}}(t)$ for $\theta=0$ and 0.2, we see that their initial values get smaller as θ increases, while the decay time of the correlation function remains practically the same. On the other hand, as θ increases to 0.5, the initial value also increases but the decay is much faster. Consequently, the denominator in the ratio of Eq. (5) decreases and the ratio

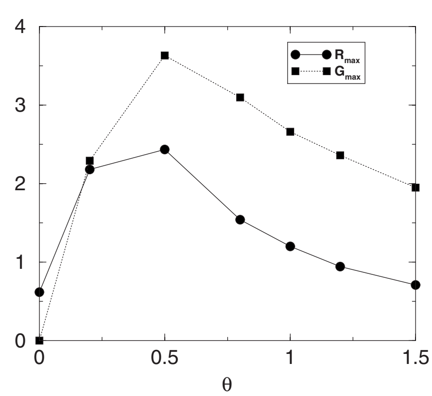

FIG. 4. Peak values of the signal-to-noise ratio ($R_{\max}$) and gain ($G_{\max}$) of an $N=10$ array of bistable units vs the coupling strength θ. The driving term is rectangular with $\Omega=0.01$ and $A=0.1$.

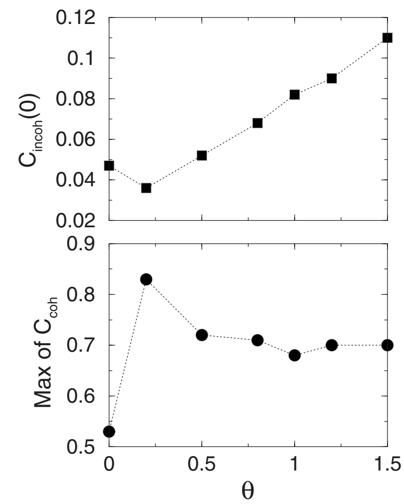

FIG. 5. The maximum value of $C_{\text{coh}}(t)$ and the initial value of the incoherent part, $C_{\text{incoh}}(0)$ for several values of θ.

reaches a maximum value. As the coupling constant is increased further, the initial values increase and therefore the values of R decrease. These features indicate that large enhancements of the SR quantifiers and large gain values are achieved when the output fluctuations are small and fast decaying. These two combined facts can only be achieved when the system operates in a nonlinear regime. For a weak driving force this nonlinear regime is not possible with a single-unit system or with an array of noninteracting units.

A reliable qualitative explanation of the just mentioned behaviors is hindered by the multidimensional character of the potential energy surface. One might try to use some assumptions to obtain a simplifying picture of the $S(t)$ dynamics. In Sec. IV we discuss some of those approximations and their shortcomings for the range of parameter values of interest to the present work. It is clear that the results presented above come up due to a complicated interplay of nonlinearity, noise forces, driving forces and the coupling strength between the individual units. When the subunits are uncoupled, the N potential energy surface is symmetrical in all directions with barriers of equal heights. Each subunit has to

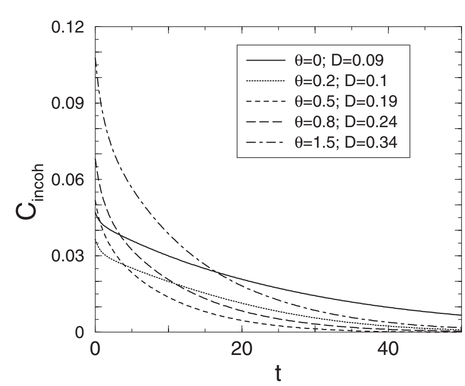

FIG. 6. Incoherent part of the correlation functions of the collective variable $S(t)$ for $N=10$ and several values of θ. The driving term is rectangular with $\Omega=0.01$ and $A=0.1$. The noise values used correspond to the values at which R reaches its peak for the different coupling strength.

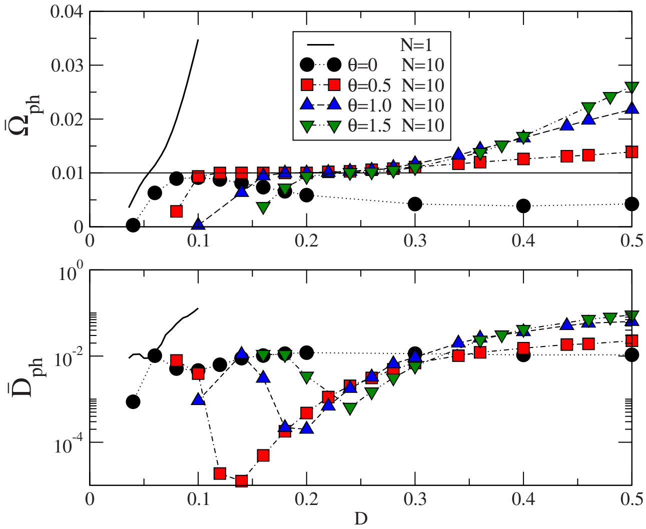

FIG. 7. (Color online) Phase frequency and phase diffusion coefficient as functions of D. The driving term is rectangular with $\Omega = 0.01$ and $A = 0.1$.

overcome the same potential barrier helped by the noise and driving forces. The reduction in the output fluctuations of the collective variable with respect to that of a single individual is simply a consequence of the central limit theorem, i.e., a size effect. Nonetheless, the output fluctuations of the collective variable are still as long lived as the fluctuations of each independent subunit. As the coupling strength is turned on, the N-dimensional energy surface is distorted. Barriers along some directions are lowered and transitions among the minima are much facilitated. Consequently, the fluctuations will no longer be long lived. This fact, together with the size effect mentioned above, leads to an increase in R_{max}. For large coupling strengths, the array becomes increasingly more rigid. Overcoming the barriers requires larger noise strengths and consequently the fluctuations increase as θ increases. One might then expect that there is an adequate value of the coupling parameter where R_{max} is maximized.

In [20] a system-size resonance effect was found in a system identical to the one treated in the present work. The origin of that system size resonance is traced back to the

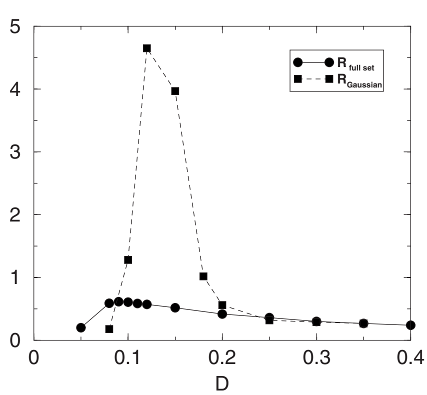

FIG. 8. Comparison of the signal-to-noise ratio as obtained from the numerical simulation of the full set of equations and from the Gaussian approximation for $N = 10$ uncoupled units ($\theta = 0$). Other parameter values are $\Omega = 0.01$ and $A = 0.1$.

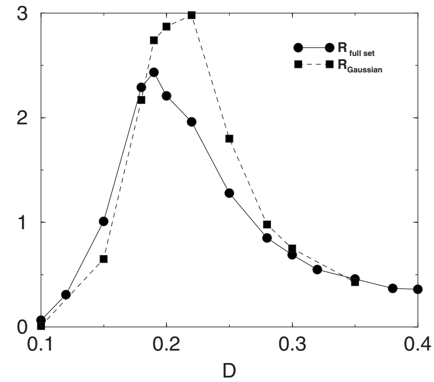

FIG. 9. Comparing the signal-to-noise ratio as obtained from the numerical simulation of the full set of equations and from the Gaussian approximation for $N = 10$ weakly ($\theta = 0.5$) coupled units. Other parameter values are $\Omega = 0.01$ and $A = 0.1$.

behavior of the linear response function. Consequently, it has the same origin as the conventional stochastic resonance for systems operating in a linear regime. As we will discuss further in Sec. IV, Pikovsky *et al.* describe the dynamics of the collective variable $S(t)$ in the Gaussian approximation. Assuming a slaving principle, they can further construct an effective Langevin equation for $S(t)$, and from here, a theoretical linear response function is obtained. For the parameter values used in [20], the theoretically derived linear response function matches pretty well the one obtained from numerical simulations, indicating that the array is operating in the linear regime. The linear response function shows a nonmonotonic behavior with the system size, analogous to its behavior with the noise strength. Consequently, the system-size resonance behavior appears in systems showing conventional stochastic resonance effects. In the present work, we are considering the same model as in [20], but we are using parameter values pertinent to a nonlinear regime dominated by a control of the output fluctuations by the driving force which can not be described by the linear response function. Nonetheless, as in the regime analyzed in [20], our results indicate that for the model system considered here, when a nonmonotonic behavior with the noise strength exists, then

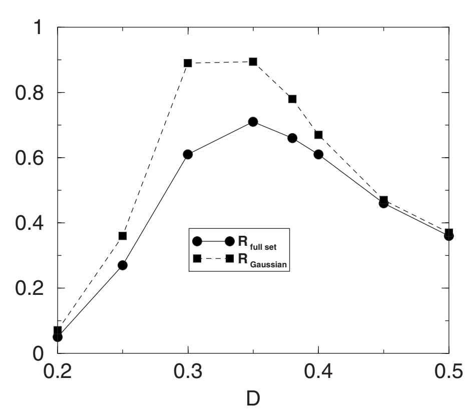

FIG. 10. Comparison of the signal-to-noise ratio as obtained from the numerical simulation of the full set of equations and from the Gaussian approximation for $N = 10$ coupled units with coupling strength $\theta = 1.5$. Other parameter values are $\Omega = 0.01$ and $A = 0.1$.

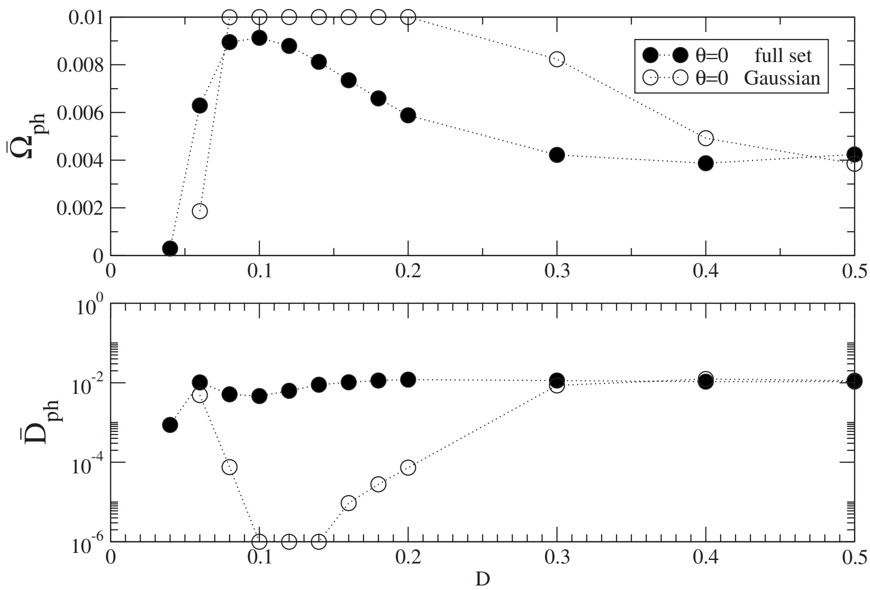

FIG. 11. Comparison of average phase frequency and phase diffusion as obtained from the numerical simulation of the full set of equations and from the Gaussian approximation for $N=10$ and $\theta=0$. Other parameter values are $\Omega=0.01$ and $A=0.1$.

nonmonotonic behaviors with other parameters like the coupling strength are expected.

Let us now turn our attention to the noise induced synchronization. In Fig. 7 the phase frequency ($\bar{\Omega}_{ph}$) and the phase diffusion coefficient ($\bar{D}_{ph}$) obtained from numerical simulations are depicted. The driving force in all cases is a rectangular input with amplitude $A=0.1$ and fundamental frequency $\Omega=0.01$. It is clear that, for this weak driving force and for a single-unit system, the range of noise values at which the phase frequency matches the driving fundamental frequency is extremely narrow to consider that there is a proper synchronization. On the other hand, for $N=10$ coupled particles, the range of noise values leading to proper frequency matching and small phase diffusion coefficients is substantial. Optimum synchronization is obtained for $\theta=0.5$ with a dip at the diffusion constant for $D\approx 0.15$. This noise value is slightly smaller than the one at which the signal-to-noise ratio reaches its maximum value for the same coupling constant ($D\approx 0.19$). Note that as the value of θ increases above 0.5, the range of noise values for synchronization to take place reduces and it shifts to higher values of D. Perhaps, the most surprising result is that for uncoupled units, synchronization between the collective variable and the driving term is lost. This is indeed in sharp contrast with the results reported in [19] for the same system with a driving force with a much larger amplitude $A=0.3$.

The range of noise values for stochastic resonance and noise induced phase synchronization do not have to coincide. Indeed, our results indicate that SR might be present at parameter values where noise-induced phase synchronization does not exist. This is not surprising. Both effects are aspects of the stochastic response of the system to a driving agent. They are undoubtedly related, but they probe different aspects of that response.

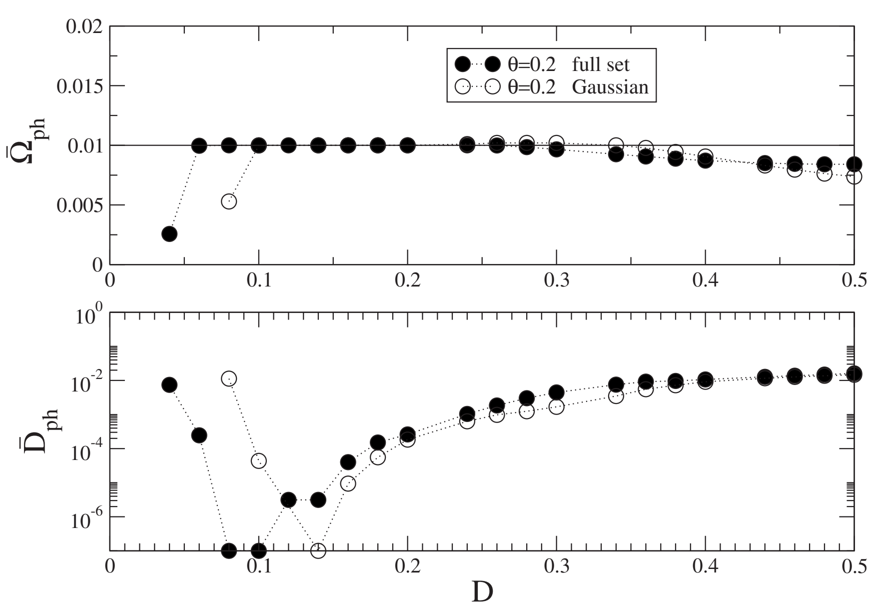

FIG. 12. Comparison of average phase frequency and phase diffusion as obtained from the numerical simulation of the full set of equations and from the Gaussian approximation for $N=10$ and $\theta=0.2$. Other parameter values are $\Omega=0.01$ and $A=0.1$.

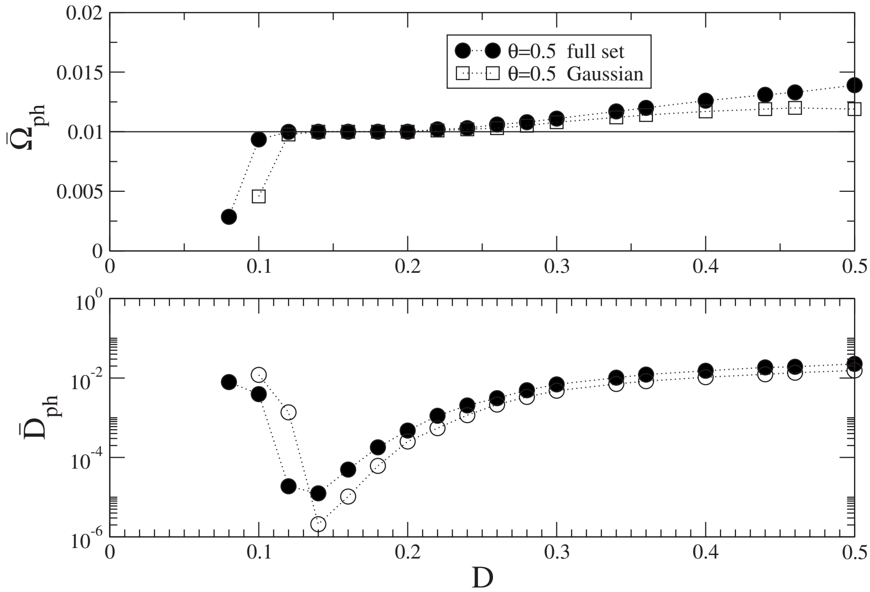

FIG. 13. Comparison of average phase frequency and phase diffusion as obtained from the numerical simulation of the full set of equations and from the Gaussian approximation for $N=10$ and $\theta=0.5$. Other parameter values are $\Omega=0.01$ and $A=0.1$.

IV. COMPARISON WITH APPROXIMATE DESCRIPTIONS

Due to the nonlinearity of the dynamical equations, the collective variable does not obey a single closed Langevin equation, but an infinite hierarchy of equations. Recently, for finite arrays, approximate descriptions of the dynamics based on truncations of the infinite hierarchy of fluctuating cumulants have been put forward by Pikovsky *et al.* [20] and by Cubero [21]. Following [20], one can write $x_i(t)=S(t)+\delta_i(t)$. The second fluctuating moment $M(t)$ is then defined by $M(t)=(1/N)\Sigma_i\delta_i^2$. Actually, as discussed in [21], one can define the fluctuating moments of all orders by $M_k(t)=(1/N)\Sigma_i\delta_i^k$, $k=1,2,\ldots$, and the corresponding fluctuating cumulants. Clearly, $M_2(t)=M(t)$. Notice that these fluctuating moments and cumulant moments are different from the usual nonfluctuating moments and cumulant moments used by Desai and Zwanzig (see [22]). In particular, notice that $M_1(t)=0$, while the typical nonfluctuating first moment is $\langle S(t)\rangle$. The Gaussian approximation in [20,21] amounts to neglecting fluctuating cumulants of order higher than two. Consequently, in the Gaussian approximation, the stochastic dynamics of the collective variable is described by the equations

$$\dot{S}=S-S^3-3MS+\sqrt{\frac{2D}{N}}\xi(t)+F(t), \qquad (14)$$

$$\frac{1}{2}\dot{M}=M-3S^2M-3M^2-\theta M+D. \qquad (15)$$

Even within the reduced Gaussian approximation, one has to rely on numerical treatments to obtain reliable information. The predictions of the Gaussian approximation (without invoking a slaving principle) are compared with those obtained from the full solution of the entire dynamics in the next figures.

In Figs. 8–10 we show the results obtained for the noise dependence of the collective signal-to-noise ratio as given by the simulation of the full set of Langevin equations and by the Gaussian approximation. In Fig. 8 we consider the case of uncoupled arrays, while in Figs. 9 and 10 we consider moderate ($\theta=0.5$) and strong ($\theta=1.5$) coupling cases, respectively. The Gaussian approximation seems to be more reliable as the value of the coupling strength increases, although it yields poor results around the noise values at which R shows its peak. Only for high values of the noise strength does the Gaussian approximation lead to results in good agreement with those obtained from the full set of equations.

As discussed in [20], the Gaussian truncation combined with a slaving principle, allows for a simplified description of the dynamics of $S(t)$ in terms of a Langevin equation in an effective double-well potential and a noise term $\sqrt{2D/N}\xi(t)$. This proposal (or the effective potential proposed in [21]) has the attractive feature of reducing the dynamics for the collective variable to a single Langevin equation in a one-dimensional potential. Unfortunately, by contrast with the parameter values for θ and D considered in those works, for the parameter values used here the effective potential evaluated as indicated in the above mentioned references might not be bounded or might not even exist.

We have also compared the results obtained with the Gaussian truncation for the average phase frequency and the phase diffusion coefficient with respect to the ones obtained from the full set of Langevin equations. In Fig. 11, we depict the results obtained for $N=10$ uncoupled units ($\theta=0$) driven by a rectangular periodic force with $A=0.1$ and fundamental frequency $\Omega=0.01$. The Gaussian approximation predicts a perfect matching of the driving frequency and the phase frequency for a wide range of noise strength values. These results are completely at variance with those obtained from the numerical solution of the full set of equations. Thus, the Gaussian approximation is not reliable for the $\theta=0$ case.

On the other hand, as depicted in Fig. 12, the introduction of even a small coupling between the units drastically changes the picture. The Gaussian approximation results are in very good agreement with those obtained from the full set of equations. Indeed, as depicted in Figs. 12–15, for coupled units the results of the Gaussian approximation are very good

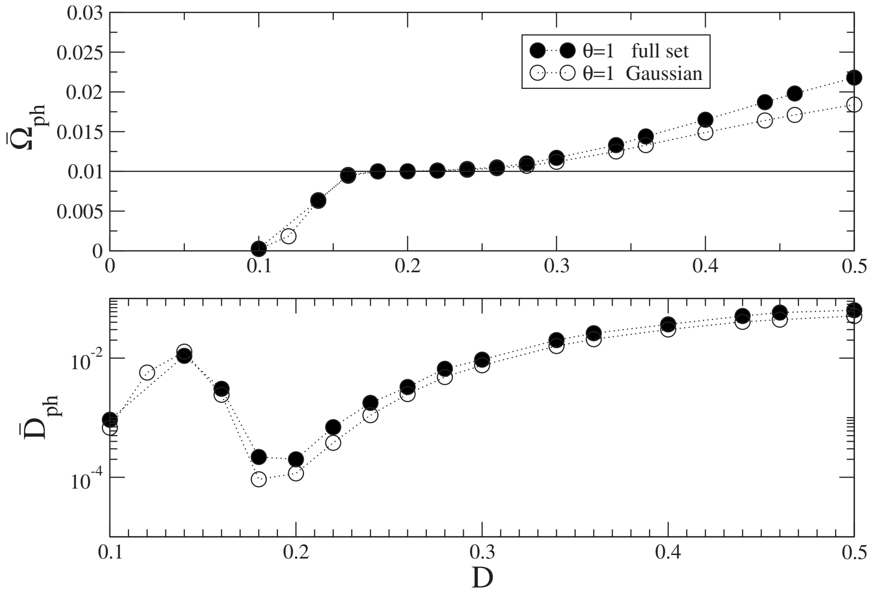

FIG. 14. Comparison of average phase frequency and phase diffusion as obtained from the numerical simulation of the full set of equations and from the Gaussian approximation for $N=10$ and $\theta=1$. Other parameter values are $\Omega=0.01$ and $A=0.1$.

compared with those obtained from the full set of dynamical equations for some range of noise values. The range of noise values for phase synchronization depends on the coupling strength. As θ is increased the range decreases. In all cases, synchronization starts disappearing as the noise strength becomes large. This is to be expected as large noise values might induce jumps over potential barriers quite independently of the driving force. Thus, the synchrony between noise-induced jumps and the changes of sign of the external amplitude tends to disappear. The very same idea of the phase process introduced previously in Eq. (9) loses its meaning for large noise strengths.

V. CONCLUDING REMARKS

We have analyzed different aspects of the stochastic collective response of a finite array of globally coupled bistable units to a weak time-periodic driving force. We focus our analysis on the phenomenon of stochastic resonance and noise induced phase synchronization. As demonstrated by our numerical results, both effects might be present in the collective response as long as the units are not statistically independent, i.e., their coupling is not zero.

There are two relevant facts: (i) the gain of the collective variable might reach values greater than unity and (ii) the phase frequency might synchronize with the fundamental driving frequency in wide ranges of the noise strength. These two features clearly indicate that, for the weak driving forces considered here, the response of the system can not be analyzed with the tools of linear response theory. This is so, even though for a single unit subject to the same weak driver, LRT provides a very valuable tool to understand SR at qualitative and even quantitative levels. Our calculations indicate that the failure of LRT is essentially due to the strong modification of the output fluctuations brought up by the external driving and the coupling between the elements of the array.

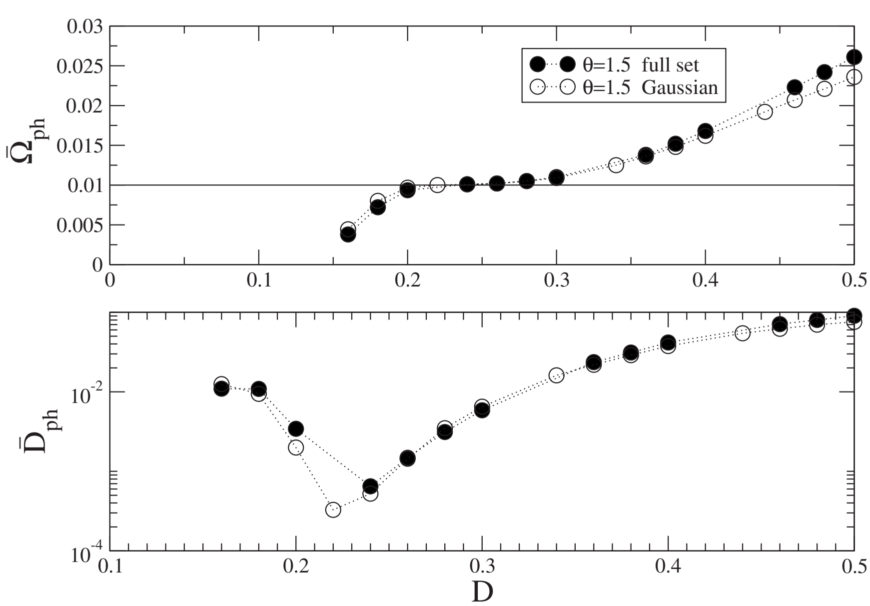

FIG. 15. Comparison of average phase frequency and phase diffusion as obtained from the numerical simulation of the full set of equations and from the Gaussian approximation for $N=10$ and $\theta=1.5$. Other parameter values are $\Omega=0.01$ and $A=0.1$.

The output fluctuations become much smaller and short lived than those present in a single-unit system.

We have also analyzed the nonmonotonic behavior of the signal-to-noise ratio with the coupling strength when other parameter values (number of particles, amplitude, and period of the driver) are kept constant. The largest signal-to-noise ratio occurs at an optimum value of the coupling strength.

Finally, we have also compared our numerical findings with those obtained by simplifying approximations that have been put forward in the literature. In particular, the closure of an infinite hierarchy of fluctuating cumulant moments at the Gaussian level provides a rather good description of the simulation results at least for some range of parameter values. Unfortunately, further simplifications leading to effective one-dimensional Langevin dynamics for the collective variable, which seems to provide useful insight on the response for some regions of parameter space, become invalid in the parameter region considered in our work.

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[1] R. Kubo, J. Phys. Soc. Jpn. **12**, 570 (1957).
[2] Peter Hänggi and Harry Thomas, Phys. Rep. **88**, 207 (1982).
[3] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[4] M. I. Dykman, R. Mannella, P. V. E. McClintock, and N. G. Stocks, Phys. Rev. Lett. **65**, 2606 (1990); **68**, 2985 (1992).
[5] J. Casado-Pascual, C. Denk, J. Gómez-Ordóñez, M. Morillo, and P. Hänggi, Phys. Rev. E **67**, 036109 (2003).
[6] J. Casado-Pascual, C. Denk, J. Gómez-Ordóñez, M. Morillo, and P. Hänggi, Phys. Rev. E **68**, 061104 (2003).
[7] Peter Jung, Ulrich Behn, Eleni Pantazelou, and Frank Moss, Phys. Rev. A **46**, R1709 (1992).
[8] A. R. Bulsara and G. Schmera, Phys. Rev. E **47**, 3734 (1993).
[9] M. Morillo, J. Gómez-Ordóñez, and J. M. Casado, Phys. Rev. E **52**, 316 (1995); J. M. Casado and M. Morillo, *ibid.* **52**, 2088 (1995).
[10] Alexander Neiman, Lutz Schimansky-Geier, and Frank Moss, Phys. Rev. E **56**, R9 (1997).
[11] John F. Lindner, Brian K. Meadows, William L. Ditto, Mario E. Inchiosa, and Adi R. Bulsara, Phys. Rev. Lett. **75**, 3 (1995).
[12] J. M. Casado, J. Gómez-Ordóñez, and M. Morillo, Phys. Rev. E **73**, 011109 (2006).
[13] D. Cubero, J. Casado-Pascual, J. Gómez-Ordóñez, J. M. Casado, and M. Morillo, Phys. Rev. E **75**, 062102 (2007).
[14] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Science* (Cambridge University Press, Cambridge, U.K., 2001).
[15] J. Kurths, S. Boccaletti, C. Grebogi, and Y. C. Lai, Chaos **13**, 126 (2003).
[16] L. A. Vainstein and D. E. Vakman, *Frequency Analysis in the Theory of Oscillations and Waves* (Nauka, Moscow, 1983).
[17] L. Schimansky-Geier, V. Anishchenko, and A. Neiman, in *Neuro-Informatics*, edited by S. Gielen and F. Moss, Handbook of Biological Physics, Vol. 4, (Elsevier Science, New York, 2001); B. Lindner, J. Garcia-Ojalvo, A. Neiman, and L. Schimansky-Geier, Phys. Rep. **392**, 321 (2004).
[18] I. Goychuk, J. Casado-Pascual, M. Morillo, J. Lehmann, and P. Hanggi, Phys. Rev. Lett. **97**, 210601 (2006).
[19] J. M. Casado, J. Gómez-Ordóñez, and M. Morillo, Europhys. Lett. **79**, 50002 (2007).
[20] A. Pikovsky, A. Zaikin, and M. A. de la Casa, Phys. Rev. Lett. **88**, 050601 (2002).
[21] David Cubero, Phys. Rev. E **77**, 021112 (2008).
[22] Rashmi C. Desai and Robert Zwanzig, J. Stat. Phys. **19**, 1 (1978).
[23] J. Casado-Pascual, J. Gómez-Ordóñez, M. Morillo, J. Lehmann, I. Goychuk, and P. Hänggi, Phys. Rev. E **71**, 011101 (2005).
[24] J. A. Freund, A. B. Neiman, and L. Schimansky-Geier, Europhys. Lett. **50**, 8 (2000).
[25] P. Talkner, Physica A **325**, 124 (2003).
[26] T. Prager and L. Schimansky-Geier, Phys. Rev. E **71**, 031112 (2005).
[27] H. S. Greenside and E. Helfand, Bell Syst. Tech. J. **60**, 1927 (1981).